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Abstract

Quadrature rules on the positive real half-line obtained by integrating the Baskakov quasi-interpolants described in [3, 6] are constructed and their asymptotic convergence orders are studied. These results are illustrated by some numerical examples. The formulas are based on series of values of the function on uniform partitions and are not comparable with Gauss quadrature rules.

Keywords: Quadrature rules, Approximation operators.
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1 Introduction

The Baskakov operators [1] are the linear approximation operators defined by

$$\mathcal{V}_n f(x) := \sum_{k\geq 0} f_k\, v_{k,n}(x), \quad f_k := f\left(\frac{k}{n}\right)$$

where the system of basic functions is

$$\mathbb{V}_n := \{v_{k,n}(x) := \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}, \quad k \geq 0\}$$

From the identities [3]

$$\mathcal{V}_n \nu_{n,k}(x) = \lambda_{n,k} m_k(x)$$

where $\nu_{n,k}(x) := \prod_{i=0}^{k-1}(x - \frac{i}{n})$, $\lambda_{n,k} := \prod_{i=0}^{k-1}(1 + \frac{i}{n})$ and $m_k(x) := x^k$, one deduces that $\mathcal{V}_n$ is an isomorphism of the space $\mathbb{P}$ of polynomials. As $\mathcal{V}_n \nu_0 = m_0$ and $\mathcal{V}_n \nu_1 = m_1$, with $\nu_0 = m_0$ and $\nu_1 = m_1$, this operator is exact on $\mathbb{P}_1$. Moreover, it has been proved in [6] that both $\mathcal{V}_n$ and $\mathcal{U}_n := \mathcal{V}_n^{-1}$ can be written as differential operators on $\mathbb{P}$ (here $D = d/dx$):

$$\mathcal{V}_n = \sum_{r\geq 0} \theta_r^{(n)}(x) D^r \qquad \mathcal{U}_n = \sum_{r\geq 0} \eta_r^{(n)}(x) D^r.$$

The polynomial coefficients $\theta_r^{(n)}(x)$ and $\eta_r^{(n)}(x)$ are in $\mathbb{P}_r$ and can be computed by recurrence relations (see Section 2 below). This allows to define the partial inverses

$$\mathcal{U}_n^{(r)} = \sum_{j=0}^{r} \eta_j^{(n)}(x) D^j, \quad 0 \leq r \leq n$$

and the (left) Baskakov *quasi-interpolants of order* r [3] for any polynomial p:

$$\mathcal{V}_n^{(r)}p(x) = \mathcal{U}_n^{(r)}\mathcal{V}_n p(x) = \sum_{j=0}^{r} \eta_j^{(n)}(x) D^j \mathcal{V}_n p(x), \quad 0 \le r \le n.$$

The aim of the present paper is to study the quadrature formulas on the real positive line obtained by integrating Baskakov quasi-interpolants. Here is an outline of the paper. The main properties of these operators, studied in [3] and [6], are recalled in Section 2. Section 3 presents the quadrature formulas (abbr. QF). Section 4 gives the quadrature weights. Section 5 contains results on convergence orders of the QF. Section 6 gives a particular case of the convergence results for a specific family of functions. Section 7 presents some numerical experiments. Finally, Section 8 gives the coefficients of QF of orders $4 \le r \le 9$. It is difficult to say if these formulas are really interesting in practice. From the numerical experiments that we have done, it is not quite clear to determine the best class of functions to which the Baskakov quadrature formulas are adapted. In a future paper, we plan to compare the latter formulas to those obtained by integration of Szász-Mirakyan quasi-interpolants [4].

Notations. $I(f) = \int_0^{+\infty} f(x)dx$, $X := x(1+x), Z := \frac{x}{1+x}$. Rising and falling factorials are denoted respectively by

$$(n)_r := n(n+1)\dots(n+r-1), \qquad [n]_r := n(n-1)\dots(n-r+1).$$

2 Baskakov quasi-interpolants

In the following, we often omit the upper index n when the latter is fixed.

Theorem 1. *The polynomials θ_r and η_r satisfy the following recurrence relations:*

$$n(r+1)\theta_{r+1}(x) = X(D\theta_r(x) + \theta_{r-1}(x)), \quad \theta_0 = 1, \theta_1 = 0.$$

$$(n+r)(r+1)\eta_{r+1}(x) = -r(1+2x)\eta_r(x) - X\eta_{r-1}(x), \quad \eta_0 = 1, \eta_1 = 0.$$

For example, the first polynomials η are as follows

$$\eta_2 = -\frac{X}{2(n+1)}, \quad \eta_3 = \frac{(1+2x)X}{3(n+1)(n+2)}, \quad \eta_4 = \frac{X((n-6)X-2)}{8(n+1)(n+2)(n+3)}$$

A more complete table is given in [6].

Theorem 2. *For all $r \ge 0$, there exists a constant $C_r > 0$ such that*

$$\|\mathcal{V}_n^{(r)}\|_\infty \le C_r \quad \forall n \ge r$$

Defining the limit polynomials

$$\bar{\eta}_{2r-1} := \lim_{n\to\infty} n^r \eta_{2r-1}^{(n)} = \frac{(-1)^r}{3\cdot 2^{r-2}(r-2)!}(1+2x)X^{r-1}, \quad \bar{\eta}_{2r} := \lim_{n\to\infty} n^r \eta_{2r}^{(n)} = \frac{(-1)^r}{2^r r!}X^r,$$

we have the following asymptotic formulas for smooth functions.

Theorem 3 (Voronovskaya type). *Let $f \in C^{2r+4}(\mathbb{R}_+)$ be a function with $D^{2r+4}f$ bounded, then for all $x \geq 0$, there holds, when $n \to \infty$*

$$\lim n^{r+1}(f(x) - \mathcal{V}_n^{(2r)} f(x)) = \bar{\eta}_{2r+1} D^{2r+1} f(x) + \bar{\eta}_{2r+2} D^{2r+2} f(x)$$

$$\lim n^{r+1}(f(x) - \mathcal{V}_n^{(2r+1)} f(x)) = \bar{\eta}_{2r+2} D^{2r+2} f(x)$$

3 Quadrature rules over the positive real halfline

By integrating $\mathcal{V}_n^{(2r)} f(x)$ and $\mathcal{V}_n^{(2r+1)} f(x)$, one obtains two quadrature formulas $QF_n^{(2r)}$ and $QF_n^{(2r+1)}$ of the same convergence order $O(n^{-(r+1)})$.
First, omitting the index n, we have, for $n \geq 2$ and for all $k \geq 0$,

$$v_k(x) := \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \Rightarrow \int_0^{+\infty} v_k(x) dx = \frac{1}{n-1},$$

therefore we get, for the classical Baskakov operators $\mathcal{V}_n = \mathcal{V}_n^{(0)} = \mathcal{V}_n^{(1)}$:

$$QF_n^{(0)}(f) = QF_n^{(1)}(f) := \int_0^{+\infty} \mathcal{V}_n f(x) dx = \frac{1}{n-1} \sum_{k \geq 0} f_k.$$

For the QI of order $r = 2$, we obtain

$$\mathcal{V}_n^{(2)} f(x) = \mathcal{V}_n f(x) + \eta_2^{(n)}(x) D^2 \mathcal{V}_n f(x)$$

$$= (1+x)^{-n} \left(\sum_{k \geq 0} f_k \binom{n+k-1}{k} Z^k - \frac{n}{2} \sum_{k \geq 0} \binom{n+k+1}{k} (\Delta^2 f_k)\, Z^{k+1} \right)$$

and by integrating we get

$$\int_0^{+\infty} \mathcal{V}_n^{(2)} f(x) dx = \frac{1}{n-1} \sum_{k \geq 0} f_k - \frac{1}{2(n^2-1)} \sum_{k \geq 0} (k+1)(n+k+1)(\Delta^2 f_k)$$

which can also be written under the form

$$QF_n^{(2)}(f) := \int_0^{+\infty} \mathcal{V}_n^{(2)} f(x) dx = \frac{1}{n-1} f_0 + \frac{n}{n^2-1} \sum_{k \geq 1} f_k.$$

In the same way, one obtains the following quadrature formulae

$$QF_n^{(3)}(f) := \int_0^{+\infty} \mathcal{V}_n^{(3)} f(x) dx$$

$$QF_n^{(3)}(f) = \frac{n-3}{6(n^2-1)} f_0 + \frac{4n^2+5n-12}{3(n+2)(n^2-1)} f_1 + \frac{n^2+2n-4}{(n+2)(n^2-1)} \sum_{k \geq 2} f_k.$$

In Section 8 are given the quadrature formulas for $4 \leq r \leq 9$.

4 Computation of quadrature weights

As $QF_n^{(r)}(f) := \sum_{k\geq 0} f_k \int_0^{+\infty} v_k^{(r)}(x)dx$, we need to compute and integrate the functions

$$v_k^{(r)}(x) := v_k(x) + \sum_{j=2}^{r} \eta_j(x) D^j v_k(x) \quad \text{for} \quad k \geq r-1$$

Setting

$$A_{k,n}^{(r)} := \int_0^{+\infty} v_{k,n}^{(r)}(x)dx, \quad k \geq 0,$$

the preceding formula provides

$$A_{k,n}^{(r)} := A_{k,n}^{(r-1)} + \int_0^{+\infty} \eta_r^{(n)}(x) D^r v_{k,n}(x).$$

In order to compute these coefficients, we integrate by parts the products $\eta_r^{(n)}(x)D^r v_{k,n}(x)$, using the derivatives at $x = 0$ of the polynomials $\eta_r^{(n)}(x)$ and of the rational functions $v_{k,n}(x)$.

4.1 Derivatives of $v_{k,n}(x)$

By direct computation, one obtains

Lemma 1. *The derivatives of basic functions $v_{k,n}(x)$ are given by*

$$Dv_{k,n}(x) = n(v_{k-1,n+1}(x) - v_{k,n+1}(x)).$$

More generally, for all $r \geq 2$

$$D^r v_{k,n}(x) = (n)_r \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} v_{k-j,n+r}(x)$$

When $x \to \infty$, $v_{k,n}(x) \sim \binom{n+k-1}{k} x^{-n}$, thus $v_{k-j,n+r}(x) \sim \binom{n+k+r-j}{k-j} x^{-(n+r)}$, therefore $D^r v_{k,n}(x) \sim \omega(n,r,k)x^{-(n+r)}$ where we set

$$\omega(n,r,k) := (n)_r \left(\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \binom{n+k+r-j}{k-j} \right).$$

On the other hand, we have

$$\eta_{2r-1}^{(n)}(x) \sim c_{2r-1} n^{-r} x^{2r-1}, \quad \eta_{2r}(x) \sim c_{2r} n^{-r} x^{2r}$$

with $c_{2r-1} =$ and $c_{2r} =$. We then deduce

$$\eta_{2r-1}^{(n)}(x) D^{2r-1} v_{k,n}(x) \sim c_{2r-1} n^{-r} x^{2r-1} \omega(n, 2r-1, k) x^{-(n+2r-1)} = O(x^{-n})$$

$$\eta_{2r}^{(n)}(x) D^{2r} v_{k,n}(x) \sim c_{2r} n^{-r} x^{2r} \omega(n, 2r, k) x^{-(n+2r)} = O(x^{-n})$$

Similarly

$$D^s\eta_r^{(n)}(x) = O(x^{r-s}), \quad D^{r-s}v_{k,n}(x) = O(x^{-(n+r-s)})$$

therefore we have also

$$D^s\eta_r(x)D^{r-s}v_{k,n}(x) = O(x^{-n})$$

This proves the following

Lemma 2. *For all n, r and $0 \leq s \leq r$, there holds*

$$\lim_{x\to+\infty} D^s\eta_r(x)D^{r-s}v_{k,n}(x) = 0$$

We also need the values of $D^s v_{k,n}(0)$.

Lemma 3. *For all n, r, there holds*

$$D^r v_{k,n}(0) = (-1)^{r-k}(n)_r\binom{r}{k}.$$

Proof. Starting from $v_{0,n}(0) = 1$ and $v_{k,n}(0) = 0$ for $k \geq 1$, we deduce $Dv_{k,n}(0) = n(v_{k-1,n+1}(0)-v_{k,n+1}(0))$, thus $Dv_{0,n}(0) = -n$, $Dv_{1,n}(0) = n$. Then $D^2v_{k,n}(0) = n(n+1)(v_{k-2,n+2}(0) - 2v_{k-1,n+2}(0) + v_{k,n+2}(0))$ implies that

$$D^2v_{0,n}(0) = (n)_2, \quad D^2v_{1,n}(0) = -2(n)_2, \quad D^2v_{2,n}(0) = (n)_2.$$

More generally, it is easy to prove by induction that

$$D^r v_{k,n}(0) = (n)_r\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}v_{k-j,n+r}(0) = (-1)^{r-k}(n)_r\binom{r}{k}. \quad \blacksquare$$

4.2 Derivatives of $\eta_r^{(n)}(x)$

Starting from the recurrence relation

$$(n+r)(r+1)\eta_{r+1}(x) = -r(1+2x)\eta_r(x) - X\eta_{r-1}(x),$$

we first deduce

$$(n+r)(r+1)\eta_{r+1}(0) = -r\eta_r(0), \quad \eta_1(0) = 0,$$

thus, we have $\eta_r(0) = 0$ for $r \geq 1$.

For first derivatives, we have $D\eta_0(0) = D\eta_1(0) = 0$ and

$$(n+r)(r+1)D\eta_{r+1}(x) = -2r\eta_r(x) - (1+2x)\eta_{r-1}(x) - r(1+2x)D\eta_r(x) - XD\eta_{r-1}(x),$$

Thus, for $r \geq 1$, we get

$$(n+r)(r+1)D\eta_{r+1}(0) = -2r\eta_r(0) - \eta_{r-1}(0) - rD\eta_r(0).$$

This gives, for $r = 1$

$$2(n+1)D\eta_2(0) = -\eta_0(0) = -1 \quad \Rightarrow \quad D\eta_2(0) = -\frac{1}{2(n+1)} = -\frac{1}{2n_2}$$

and for $r = 2$

$$3(n+2)D\eta_3(0) = -2D\eta_2(0) \quad \Rightarrow \quad D\eta_3(0) = \frac{1}{3(n+1)(n+2)} = \frac{1}{3n_3}$$

More generally, assuming that $D\eta_r(0) = (-1)^{r-1}/rn_r$ for $r \geq 3$, we obtain

$$(n+r)(r+1)D\eta_{r+1}(0) = -rD\eta_r(0) \quad \Rightarrow \quad D\eta_{r+1}(0) = \frac{(-1)^r}{(r+1)n_{r+1}}$$

For second derivatives, we have $D^2\eta_0(0) = D^2\eta_1(0) = 0$ and

$$(n+r)(r+1)D^2\eta_{r+1}(x) = -4rD\eta_r(x) - 2\eta_{r-1}(x) - 2(1+2x)D\eta_{r-1}(x)$$
$$-r(1+2x)D^2\eta_r(x) - XD^2\eta_{r-1}(x)$$

whence

$$(n+r)(r+1)D^2\eta_{r+1}(0) = -2\eta_{r-1}(0) - 4rD\eta_r(0) - 2D\eta_{r-1}(0) - rD^2\eta_r(0),$$

that gives in particular

$$2(n+1)D^2\eta_2(0) = -2 \quad \Rightarrow \quad D^2\eta_2(0) = -\frac{1}{n_2}$$
$$3(n+2)D^2\eta_3(0) = \frac{6}{n+1} \quad \Rightarrow \quad D^2\eta_3(0) = \frac{2}{n_3}.$$

The general formula is more complex than for first derivatives. It is easy to prove by induction that for $s \geq 2$, there holds

$$(r+1)(n+r)D^s\eta_{r+1}(x) = -s(s-1)D^{s-2}\eta_{r-1}(x) - 2srD^{s-1}\eta_r(x) - s(1+2x)D^{s-1}\eta_{r-1}(x)$$
$$-r(1+2x)D^s\eta_r(x) - XD^s\eta_{r-1}(x)$$

that gives, for derivatives at $x = 0$:

$$(r+1)(n+r)D^s\eta_{r+1}(0) = -s(s-1)D^{s-2}\eta_{r-1}(0) - 2srD^{s-1}\eta_r(0) - sD^{s-1}\eta_{r-1}(0) - rD^s\eta_r(0)$$

This aloows to compute the table of derivatives at $x = 0$ of the polynomials $n_r\eta_r(x)$. We remind the notation $n_r := (n+1)\cdots(n+r)$.

r	1	2	3	4
$n_2\eta_2$	-1/2	-1		
$n_3\eta_3$	1/3	2	4	
$n_4\eta_4$	-1/4	$(n-8)/4$	$3(n-6)/2$	$3(n-6)$
$n_5\eta_5$	1/5	$-(n-6)/3$	$-4(n-3)$	$-4(5n-12)$
$n_6\eta_6$	-1/6	$(13-72)/36$	$-(n^2-46n+120)/8$	$-(9n^2-284n+480)/6$
$n_7\eta_7$	1/7	$-(11n-60)/30$	$(5n^2-142n+360)/20$	$5n^2-76n+120$

r	5	6	7
$n_5\eta_5$	$-8(5n-12)$		
$n_6\eta_6$	$-5(3n^2-86n+120)/2$	$-5(3n^2-86n+120)$	
$n_7\eta_7$	$(9n^2-106n+120)/5$	$(35n^2-378n+360)/6$	$(35n^2-378n+360)/12$

4.3 Integration of $\eta_r(x)D^r v_k(x)$

Coming back to the integral, integrating by parts and using the above lemmas, we obtain successively

$$\int_0^{+\infty} \eta_r^{(n)}(x)D^r v_{k,n}(x)dx = -\int_0^{+\infty} D\eta_r^{(n)}(x)D^{r-1}v_{k,n}(x)dx$$

$$= D\eta_r^{(n)}(0)D^{r-2}v_{k,n}(0) + \int_0^{+\infty} D^2\eta_r^{(n)}(x)D^{r-2}v_{k,n}(x)dx$$

$$= D\eta_r^{(n)}(0)D^{r-2}v_{k,n}(0) - D^2\eta_r^{(n)}(0)D^{r-3}v_{k,n}(0) - \int_0^{+\infty} D^3\eta_r^{(n)}(x)D^{r-3}v_{k,n}(x)dx$$

$$= \sum_{s=1}^{r-1}(-1)^{s-1}D^s\eta_r^{(n)}(0)D^{r-s-1}v_{k,n}(0) + (-1)^r \int_0^{+\infty} D^r\eta_r^{(n)}(x)v_{k,n}(x)dx.$$

As $\eta_r^{(n)}(x) \in \mathbb{P}_r$, $D^r\eta_r^{(n)}(x)$ is a constant which can also be computed by induction on r from the recurrence relation. Setting $\eta_r(x) = a_r x^r + \cdots$, we will have $D^r\eta_r^{(n)}(x) = r!a_r$. The leading coefficients (depending on n) satisfy, for $r \geq 1$, the recurrence relation

$$(r+1)(n+r)a_{r+1} = -2ra_r - a_{r-1}, \quad \text{with} \quad a_0 = 1, a_1 = 0.$$

Using this relation or the table above, their first values are respectively

$$a_2 = -\frac{1}{n_1}, \quad a_3 = \frac{2}{3n_2}, \quad a_4 = \frac{(n-6)}{8n_3}, \quad a_5 = -\frac{(5n-12)}{15n_4}.$$

Finally, taking into account $\int_0^{+\infty} v_{k,n}(x)dx = \frac{1}{n-1}$, we obtain

Lemma 4.

$$\int_0^{+\infty} \eta_r^{(n)}(x)D^r v_{k,n}(x)dx = \sum_{s=1}^{r-1}(-1)^{s-1}D^s\eta_r^{(n)}(0)D^{r-s-1}v_{k,n}(0) + \frac{(-1)^r r!a_r}{n-1}.$$

This method provides the coefficients of quadrature formulas given in Section 8.

Example. For $r = 4$, we have

$$A_{k,n}^{(4)} := A_{k,n}^{(3)} + \int_0^{+\infty} \eta_4^{(n)}(x)D^4 v_{k,n}(x),$$

with

$$\int_0^{+\infty} \eta_4^{(n)}(x)D^4 v_{k,n}(x)dx = \sum_{s=1}^{3}(-1)^{s-1}D^s\eta_4^{(n)}(0)D^{3-s}v_{k,n}(0) + \frac{3(n-6)}{(n-1)n_3}.$$

We now introduce the following notations.
$\gamma_{1,k} := D\eta_4^{(n)}(0)D^2 v_{k,n}(0) = 0$ for $k \geq 3$ and

$$\gamma_{1,0} = -\frac{(n)_2}{4n_3}, \quad \gamma_{1,1} = \frac{(n)_2}{2n_3}, \quad \gamma_{1,2} = -\frac{(n)_2}{4n_3}.$$

$\gamma_{2,k} := D^2\eta_4^{(n)}(0)Dv_{k,n}(0) = 0$ for $k \geq 2$ and

$$\gamma_{2,0} = -\frac{n(n-8)}{4n_3}, \quad \gamma_{2,1} = \frac{n(n-8)}{4n_3}.$$

$\gamma_{3,k} := D^3\eta_4^{(n)}(0)v_{k,n}(0) = 0$ for $k \geq 1$ and $\gamma_{3,0} = \frac{3(n-6)}{2n_3}$.

Therefore, starting from

$$A_{0,n}^{(3)} = \frac{n-3}{6(n^2-1)}, \quad A_{1,n}^{(3)} = \frac{4n^2+5n-12}{3(n+2)(n^2-1)}, \quad A_{2,n}^{(3)} = \frac{n^2+2n-4}{(n+2)(n^2-1)}.$$

we get the successive coefficients of $QF_n^{(4)}(f)$

$$A_{0,n}^{(4)} = A_{0,n}^{(3)} + \gamma_{1,0} - \gamma_{2,0} + \gamma_{3,0} + \frac{3(n-6)}{(n-1)n_3} = \frac{2n^2-11n-48}{(n-1)n_2}$$

$$A_{1,n}^{(4)} = A_{1,n}^{(3)} + \gamma_{1,1} - \gamma_{2,1} + \frac{3(n-6)}{(n-1)n_3} = \frac{19n^3+95n^2+18n-360}{4(n-1)n_3}$$

$$A_{2,n}^{(4)} = A_{2,n}^{(3)} + \gamma_{1,2} + \frac{3(n-6)}{(n-1)n_3} = \frac{3n^3+20n^2+21n-120}{4(n-1)n_3}$$

$$A_{3,n}^{(4)} = A_{3,n}^{(3)} + \frac{3(n-6)}{(n-1)n_3} = \frac{n^3+5n^2+5n-30}{(n-1)n_3}$$

5 Convergence order

5.1 General results

Assuming that f satisfies the conditions of Theorem 3 (Voronovskaja) in Section 2 and gives rise to a convergent series $S_n = \sum_{k\geq 0} f_k$, we then deduce

Theorem 4. *The following limits hold, for all $r \geq 0$,*

$$\lim n^{r+1}(I(f) - QF_n^{(2r)}(f)) = \frac{(-1)^{r+1}}{3\cdot 2^{r-1}(r-1)!}\int_0^{+\infty}(1+2x)X^rD^{2r+1}f(x)dx$$
$$+\frac{(-1)^{r+1}}{2^{r+1}(r+1)!}\int_0^{+\infty}X^{r+1}D^{2r+2}f(x)dx$$
$$\lim n^{r+1}(I(f) - QF_n^{(2r+1)}(f)) = \frac{(-1)^{r+1}}{2^{r+1}(r+1)!}\int_0^{+\infty}X^{r+1}D^{2r+2}f(x)dx$$

Therefore, we get the asymptotic convergence orders:

$$I(f) - QF_n^{(2r)} = 0(n^{-(r+1)}), \qquad I(f) - QF_n^{(2r+1)} = 0(n^{-(r+1)}).$$

By integrating by parts the right hand sides of the above limits, one obtains

Lemma 5. *The two following identities hold*

$$\int_0^{+\infty}(1+2x)X^r f^{(2r+1)}(x)dx = -2(2r+1)!I(f)+\sum_{p=0}^{r}(-1)^{r-p+1}\frac{(r+p+1)!}{r-p+1}\binom{r}{p}f^{(r-p)}(0).$$

$$\int_0^{+\infty}X^{r+1}f^{(2r+2)}(x)dx = (2r+2)!I(f)+\sum_{p=0}^{r}(-1)^{r-p}(r+p+1)!\binom{r+1}{p}f^{(r-p)}(0).$$

Proof. We only prove the second identity, that of the first being similar. Setting $g(x) := X^{r+1} = x^{r+1}(1+x)^{r+1}$, integrating by parts and using the fact that $g^{(k)}(0) = 0$ for $0 \le k \le r$, we first get

$$\int_0^{+\infty} g\cdot f^{(2r+2)} = (-1)^{r+1}\int_0^{+\infty} g^{(r+1)}\cdot f^{(r+1)} = (-1)^r g^{(r+1)}(0)f^{(r+1)}(0)+(-1)^r\int_0^{+\infty} g^{(r+2)}\cdot f^{(r)}.$$

More generally, on the one hand using the following integration by parts for $0 \le p \le r$

$$(-1)^{r-p+1}\int_0^{+\infty} g^{(r+p+1)}\cdot f^{(r-p+1)} = (-1)^{r-p}g^{(r+p+1)}(0)f^{(r-p)}(0)+(-1)^{r-p}\int_0^{+\infty} g^{(r+p+2)}\cdot f^{(r-p)}$$

and on the second hand the derivatives $g^{(r+p)}(0) = (r+p)!\binom{r}{p}$, we obtain the desired formula. To be more specific, the last step, for $p = r$, gives

$$-\int_0^{+\infty} g^{(2r+1)}\cdot f' = (r+1)(2r+1)!f(0)+\int_0^{+\infty} g^{(2r+2)}\cdot f = (r+1)(2r+1)!f(0)+(2r+2)!I(f)$$

The first integral can be written in the form

$$\int_0^{+\infty} h(x)f^{(2r+1)}(x)dx, \quad \text{with} \quad h(x) := (1+2x)X^r = x^{r+1}(1+x)^r + x^r(1+x)^{r+1}.$$

In that case, one has $h^{(k)}(0) = 0$ for $0 \le k \le r-1$, $h^{(r+p)}(0) = \frac{(r+p+1)!}{r-p+1}\binom{r}{p}$ for $0 \le p \le r$ and $h^{(2r+1)}(0) = 2(2r+1)!$, whence the first identity. ■.

Corollary 1. *The results of Theorem 4 and Lemma 5 can be written as follows*

$$\lim n^{r+1}(I(f) - QF_n^{(2r)}(f)) = (-1)^{r+1}(c_0^{(2r)}I(f) + \sum_{p=0}^{r} a_p^{(2r)} D^p f(0))$$

$$\lim n^{r+1}(I(f) - QF_n^{(2r+1)}(f)) = (-1)^{r+1}(c_0^{(2r+1)}I(f) + \sum_{p=0}^{r} a_p^{(2r+1)} D^p f(0))$$

where

$$c_0^{(2r)} = \frac{(-1)^r(4r-3)}{3\cdot 2^r r!}(2r+1)!, \quad c_0^{(2r+1)} = \frac{(-1)^{r+1}}{2^{r+1}(r+1)!}(2r+2)!$$

and, for $0 \le i \le r$,

$$a_p^{(2r)} = \frac{(-1)^p(4r-3)}{3\cdot 2^{r+1}}\frac{(r+p+1)!}{p!(r-p+1)!}, \quad a_i^{(2r+1)} = \frac{(-1)^{p+1}}{2^{r+1}}\frac{(r+p+1)!}{p!(r-p+1)!}$$

Noticing that for all p

$$c_0^{(2r)}/c_0^{(2r+1)} = a_p^{(2r)}/a_p^{(2r+1)} = -\frac{1}{3}(4r-3),$$

we deduce the following

Corollary 2. *The extrapolation formula*

$$EQF_n^{(2r+1)}(f) = \frac{3}{4r} QF_n^{(2r)} + \left(1 - \frac{3}{4r}\right) QF_n^{(2r+1)}$$

satisfies

$$\lim n^{r+1}(I(f) - EQF_n^{2r+1}(f)) = 0.$$

5.2 The family of functions $f(x) = (1+x)^{-\alpha}$

In particular, taking functions of the family $f_\alpha(x) = (1+x)^{-\alpha}$, with $\alpha > 1$, for which $I(f) = \int_0^{+\infty} f(x)dx = 1/(\alpha - 1)$, it is possible to prove the following more specific results.

Corollary 3.

$$\lim n^{r+1}(I(f_\alpha) - QF_n^{(2r)}(f_\alpha)) = \frac{(-1)^r}{3 \cdot 2^{r+1}} \frac{(4r-3)\Gamma(\alpha+2r+2)}{(\alpha-1)\Gamma(\alpha+r+1)}$$

$$\lim n^{r+1}(I(f_\alpha) - QF_n^{(2r+1)}(f_\alpha)) = \frac{(-1)^{r+1}}{2^{r+1}} \frac{\Gamma(\alpha+2r+2)}{(\alpha-1)\Gamma(\alpha+r+1)}.$$

Proof. The result is a direct consequence of corollary 1 above and the computation of derivatives

$$D^p f_\alpha(x) = (-1)^p (\alpha)_p (1+x)^{-(\alpha+p)} \quad \Rightarrow \quad D^p f_\alpha(0) = (-1)^p (\alpha)_p.$$

6 Numerical examples

In all examples given below (except in Example 3 where the series are computed via the Zeta Function), we approximate the series $\sum_{k=r}^{+\infty} f_k$ by finite sums $\sum_{k=r}^{N} f_k$,where $N = pn$ for some p large enough, depending on the example. The convergence may be slow, therefore it is sometimes necessary to use some convergence acceleration method [2] for the precise calculation of those sums.

Example 1

$$f(x) = \frac{\exp(-x)}{100+2x}, \quad I(f) = \int_0^{+\infty} f(x)dx = 9.807E-3$$

The following results are obtained by taking $N = 24n$ terms of the series. They are quite the same by taking $N = 32n$.

$n\backslash r$	0	2	3	4	5	6	7	8	9
8	2.1(-3)	2.5(-4)	-5.4(-4)	-4.4(-4)	-1.0(-4)	1.2(-4)	1.6(-4)	9.8(-5)	1.4(-5)
16	9.9(-4)	6.1(-5)	-1.6(-4)	-7.1(-5)	7.2(-6)	2.4(-5)	1.2(-5)	-5.9(-7)	-5.0(-6)
32	4.8(-4)	1.5(-5)	-4.2(-5)	-1.0(-5)	3.3(-6)	2.7(-6)	3.0(-7)	-4.3(-7)	-2.6(-7)
64	2.3(-4)	3.8(-6)	-1.1(-5)	-1.4(-6)	6.3(-7)	2.2(-7)	-2.2(-8)	-3.0(-8)	-4.4(-9)
128	1.2(-4)	9.6(-7)	-2.8(-6)	-1.8(-7)	9.5(-8)	1.6(-8)	-3.5(-9)	-1.4(-9)	-6.0(-11)
256	5.8(-5)	2.4(-7)	-7.1(-7)	-2.3(-8)	1.3(-8)	1.1(-9)	-3.0(-10)	-5.1(-11)	6.7(-12)

As predicted by Lemma 5, the derivatives at the origin being small, we observe an alternance of signs on the errors for each pair $(2r, 2r+1)$. We compute extrapolated values in the following table: the numerical convergence order (nco) tends to $r+2$ for the pair $(2r, 2r+1)$.

n	Trap	2-3	nco	4-5	nco	6-7	nco	8-9	nco
8	-1.3(-5)	5.1(-5)		-2.3(-4)		1.5(-4)		3.0(-5)	
16	-3.3(-6)	7.1(-6)	2.8	-2.2(-5)	3.4	1.5(-5)	3.3	-4.2(-6)	2.8
32	-8.3(-7)	9.3(-7)	2.9	-1.8(-6)	3.6	9.0(-7)	4.0	-2.9(-7)	3.8
64	-2.1(-7)	1.2(-7)	2.9	-1.3(-7)	3.8	3.9(-8)	4.5	-9.2(-9)	5.0
128	-5.2(-8)	1.5(-8)	3.0	-8.6(-9)	3.9	1.5(-9)	4.7	-2.0(-10)	5.4
256	-1.3(-8)	1.9(-9)	3.0	-5.6(-10)	3.9	5.0(-11)	4.9	-4.1(-12)	5.7

However, the results obtained by Romberg extrapolation from the sequence of values computed by the trapezoidal formula, are much better than the above ones. One has to take formulas of the same order, so column (8-9) above must be compared with column 3 below.
Taking $N = 16n$ (in the first table, resp. $N = 24n$ in the second) terms of the sum, we get the following tables:

$n\backslash p$	1	2	3
8	1.3(-5)		
16	3.3(-6)	7.8(-11)	
32	8.3(-7)	7.7(-10)	8.3(-10)
64	2.1(-7)	8.3(-10)	8.3(-10)
128	5.1(-8)	8.4(-10)	8.3(-10)
256	1.2(-8)	8.4(-10)	8.3(-10)

$n\backslash p$	1	2	3
8	1.3(-5)		
16	3.3(-6)	9.0(-10)	
32	8.3(-7)	5.6(-11)	2.3(-13)
64	2.1(-7)	3.3(-12)	2.5(-13)
128	5.2(-8)	3.1(-14)	2.5(-13)
256	1.3(-8)	2.4(-13)	2.5(-13)

Columns 4 to 7 contain the same numbers as column 3.
Now, taking $N = 32n$, we get

$n\backslash p$	1	2	3
8	1.3(-5)		
16	3.3(-6)	9.0(-10)	
32	8.3(-7)	5.6(-11)	7.6(-17)
64	2.1(-7)	3.5(-12)	7.6(-17)
128	5.2(-8)	2.2(-13)	7.6(-17)
256	1.3(-8)	1.4(-14)	7.6(-17)

Once again, the precision does not increase in columns 4 to 7.

Example 2

$$f(x) = \frac{\exp(-x)}{1+x^4}, \quad I(f) = \int_0^{+\infty} f(x)dx = .630477834918498$$

Here, we take $N = 16n$ for all n, except for $n = 1024$ for which we take $N = 24n$.

n	2	3	4	5	6	7	8	9
8	1.9(-2)	-4.2(-2)	-3.4(-2)	-8.1(-3)	9.0(-3)	1.2(-2)	7.7(-3)	1.1(-3)
16	4.8(-3)	-1.2(-2)	-5.6(-3)	5.5(-4)	1.9(-3)	9.5(-4)	-4.5(-5)	-3.9(-4)
32	1.2(-3)	-3.3(-3)	-8.0(-4)	2.6(-4)	2.1(-4)	2.4(-5)	-3.4(-5)	-2.0(-5)
64	3.0(-4)	-8.5(4)	-1.1(-4)	4.9(-5)	1.8(-5)	-1.7(-6)	-2.4(-6)	-3.4(-7)
128	7.4(-5)	-2.2(-4)	-1.4(-5)	7.4(-6)	1.3(-6)	-2.7(-7)	-1.0(-7)	4.7(-9)
256	1.9(-5)	-5.5(-5)	-1.8(-6)	1.0(-6)	8.5(-8)	-2.3(-8)	-3.9(-9)	5.4(-10)
512	4.6(-6)	-1.4(-5)	-2.3(-7)	1.3(-7)	5.5(-9)	-1.7(-9)	-1.4(-10)	2.3(-11)
1024	1.2(-6)	-3.5(-6)	-2.9(-8)	1.7(-8)	3.5(-10)	-1.1(-10)	-4.4(-12)	9.1(-13)

The alternance of signs still occurs for n large enough.
The table below gives the errors on extrapolated values: the numerical convergence order (nco) tends to $r+2$ for the pair $(2r, 2r+1)$.

n	Trap	2-3	nco	4-5	nco	6-7	nco	8-9	nco
8	1.3(-3)	4.0(-3)		-1.8(-2)		1.2(-2)		2.3(-3)	
16	3.2(-4)	5.5(-4)	2.9	1.7(-3)	3.4	1.2(-3)	3.3	-3.2(-4)	2.8
32	8.1(-5)	7.3(-5)	2.9	-1.4(-4)	3.6	7.0(-5)	4.1	-2.2(-5)	3.9
64	2.0(-5)	9.4(-6)	2.9	-1.0(-5)	3.8	3.1(-6)	4.5	-7.2(-7)	4.9
128	5.1(-6)	1.2(-6)	3.0	-6.7(-7)	3.9	1.2(-7)	4.7	-1.6(-8)	5.5
256	-1.3(-6)	1.5(-7)	3.0	-4.3(-8)	4.0	3.9(-9)	4.9	-3.0(-10)	5.7

Example 3

$$f(x) = \frac{1}{(1+x)^{\alpha+1}}, \quad \alpha = 1/2 \quad I(f) = 2$$

n	2	3	4	5	6	7	8	9
16	1.0(-2)	-2.6(-2)	-1.2(-2)	1.2(-3)	4.1(-3)	2.0(-3)	-9.4(-5)	-8.4(-4)
32	2.6(-3)	-7.1(-3)	-1.7(-3)	5.6(-4)	4.5(-4)	5.2(-5)	-7.4(-5)	-4.3(-5)
64	6.4(-4)	-1.8(-3)	-2.4(-4)	1.1(-4)	3.8(-5)	-3.7(-6)	-5.1(-6)	-7.4(-7)
128	1.6(-4)	-4.7(-4)	-3.1(-5)	1.6(-5)	2.7(-6)	-5.8(-7)	-2.3(-7)	1.0(-8)
256	4.0(-5)	-1.2(-4)	-3.9(-6)	2.2(-6)	1.8(-7)	-5.0(-8)	-8.5(-9)	1.2(-9)
512	1.0(-5)	-3.0(-5)	-4.9(-7)	2.9(-7)	1.2(-8)	-3.6(-9)	-2.9(-10)	-5.4(-11)
1024	2.5(-6)	-7.5(-6)	-6.2(-8)	3.7(-8)	7.6(-10)	-2.4(-10)	-9.5(-12)	2.0(-12)

Notice some instability in colums 7 and 9, also visible in numerical convergence orders below.

n	2	3	4	5	6	7	8	9
16/32	2.0	1.88	2.78	1.09	3.17	5.30	0.34	4.28
32/64	2.0	1.94	2.88	2.39	3.58	3.81	3.85	5.87
64/128	2.0	1.97	2.94	2.73	3.79	2.67	4.47	6.20
128/256	2.0	1.98	2.97	2.87	3.89	3.53	4.74	3.10
256/512	2.0	1.99	2.98	2.94	3.95	3.79	4.87	4.44
512/1024	2.0	2.0	2.99	2.97	3.97	3.9	4.9	4.76

Extrapolated values: the numerical convergence order (nco) tends fastly to $r+2$ for the pair $(2r, 2r+1)$.

n	Trap	2-3	nco	4-5	nco	6-7	nco	8-9	nco
64	3.0(-5)	2.0(-5)	3.0	-2.2(-5)	3.8	6.7(-6)	4.5	-1.6(-6)	5.0
128	7.6(-6)	2.6(-6)	2.9	-1.5(-6)	3.9	2.5(-7)	4.7	-3.5(-8)	5.5
256	1.9(-6)	3.3(-7)	3.0	-9.4(-8)	3.9	8.5(-9)	4.9	-6.5(-10)	5.8
512	4.8(-7)	4.1(-8)	3.0	-6.0(-9)	4.0	2.8(-10)	4.9	-1.1(-11)	5.9
1024	1.2(-7)	5.2(-9)	3.0	-3.8(-10)	4.0	8.9(-12)	5.0	-1.8(-13)	5.9

However, the results obtained by Romberg extrapolation from the sequence of values computed by the trapezoidal formula, are much better than the above ones. One has to take formulas of the same order, so column (8-9) above must be compared with column 3 below.

$n\backslash p$	1	2	3	4	5	6	7
16	4.9(-4)						
32	1.2(-4)	6.9(-8)					
64	3.0(-5)	4.3(-9)	9.9(-12)				
128	7.6(-6)	2.7(-10)	1.6(-13)	7.4(-16)			
256	1.9(-6)	1.7(-11)	2.4(-15)	2.9(-18)	2.3(-20)		
512	4.8(-7)	1.1(-12)	3.8(-17)	1.1(-20)	2.2(-23)	2.6(-25)	
1024	1.2(-7)	6.6(-14)	6.0(-19)	4.4(-23)	2.2(-26)	6.6(-29)	1.1(-30)

7 Quadrature formulae for $4 \le r \le 9$

For the reader's convenience, we list the coefficients of quadrature formulas for $4 \le r \le 9$. In spite of the fact that they have been checked several times, an error is always possible.

Formula $QF_n^{(4)}$

$$QF_n^{(4)}(f) := \int_0^{+\infty} \mathcal{V}_n^{(4)} f(x)dx = \sum_{j=0}^{2} A_j^{(4)}(n) f\left(\frac{j}{n}\right) + A_3^{(4)}(n) \sum_{k\ge 3} f\left(\frac{k}{n}\right)$$

with

$$A_0^{(4)}(n) = \frac{2n^2 - 11n - 48}{12(n^2-1)(n+2)}, \quad A_1^{(4)}(n) = \frac{19n^3 + 95n^2 + 18n - 360}{12(n^2-1)(n+2)(n+3)}$$

$$A_2^{(4)}(n) := \frac{3n^3 + 20n^2 + 21n - 120}{4(n^2-1)(n+2)(n+3)}, \quad A_3^{(4)}(n) := \frac{n^3 + 5n^2 + 5n - 30}{(n^2-1)(n+2)(n+3)}$$

Formula $QF_n^{(5)}$

$$QF_n^{(5)}(f) := \sum_{j=0}^{3} A_j^{(5)}(n) f\left(\frac{j}{n}\right) + A_4^{(5)}(n) \sum_{k\geq 4} f\left(\frac{k}{n}\right)$$

$$A_0^{(5)}(n) = \frac{6n^3 + 13n^2 + 143n - 480}{20(n^2-1)(n+2)(n+3)}$$

$$A_1^{(5)}(n) = \frac{91n^4 + 927n^3 + 2954n^2 - 72n - 12960}{60(n^2-1)(n+2)(n+3)(n+4)}$$

$$A_2^{(5)} = \frac{(29n^4 + 288n^3 + 1531n^2 + 2052n - 12960}{60(n+2)(n+3)(n+4)(n^2-1)}$$

$$A_3^{(5)} = \frac{6n^4 + 47n^3 + 124n^2 + 148n - 1080}{5(n+2)(n+3)(n+4)(n^2-1)}$$

$$A_4^{(5)}(n) = \frac{n^4 + 9n^3 + 25n^2 + 30n - 216}{(n+2)(n+3)(n+4)(n^2-1)}$$

Formula $QF_n^{(6)}$

$$(6) \quad QF_n^{(6)}(f) := \sum_{j=0}^{4} A_j^{(6)}(n) f\left(\frac{j}{n}\right) + A_5^{(6)}(n) \sum_{k\geq 5} f\left(\frac{k}{n}\right)$$

with $d_6 := (n+5)(n+4)(n+3)(n+2)(n^2-1)$ and

$$A_0^{(6)}(n) = \frac{1}{360d_6}((n+5)(133n^4 + 1311n^3 + 2792n^2 - 13536n - 56160))$$

$$A_1^{(6)}(n) = \frac{1}{60d_6}(81n^5 + 1212n^4 + 7429n^3 + 21698n^2 + 4920n - 100800)$$

$$A_2^{(6)}(n) := \frac{1}{120d_6}(53n^5 + 496n^4 + 1987n^3 + 17984n^2 + 50160n - 201600)$$

$$A_3^{(6)}(n) = \frac{1}{180d_6}(271n^5 + 3602n^4 + 14309n^3 + 24118n^2 + 63720n - 302400)$$

$$A_4^{(6)}(n) = \frac{1}{6d_6}(5n^5 + 79n^4 + 415n^3 + 845n^2 + 2190n - 10080)$$

$$A_5^{(6)}(n) = \frac{1}{d_6}(n^5 + 14n^4 + 70n^3 + 140n^2 + 364n - 1680)$$

Formula $QF_n^{(7)}$

$$(7) \quad QF_n^{(7)}(f) := \sum_{j=0}^{5} A_j^{(7)}(n) f\left(\frac{j}{n}\right) + A_6^{(7)}(n) \sum_{k\geq 6} f\left(\frac{k}{n}\right)$$

with $d_7 := (n+6)(n+5)(n+4)(n+3)(n+2)(n^2-1)$

$$A_0^{(7)}(n) = \frac{1}{504d_7}(n+6)(173n^5 + 2920n^4 + 20083n^3 + 56096n^2 - 69264n - 574560)$$

$$A_1^{(7)}(n) = \frac{1}{210d_7}(283n^6 + 5375n^5 + 41830n^4 + 198530n^3 + 641052n^2 + 490320n - 3024000)$$

$$A_2^{(7)}(n) = \frac{1}{840d_7}(389n^6 + 6490n^5 + 18475n^4 - 72250n^3 + 769896n^2 + 4786560n - 12096000)$$

$$A_3^{(7)}(n) = \frac{1}{315d_7}(541n^6 + 11330n^5 + 85520n^4 + 227530n^3 + 209724n^2 + 1531440n - 4536000)$$

$$A_4^{(7)}(n) = \frac{1}{35d_7}(17n^6 + 405n^5 + 4275n^4 + 18665n^3 + 28573n^2 + 177630n - 504000)$$

$$A_5^{(7)}(n) = \frac{1}{7d_7}(8n^6 + 149n^5 + 1103n^4 + 3935n^3 + 5462n^2 + 35256n - 100800)$$

$$A_6^{(7)}(n) = \frac{1}{d_7}(n^6 + 20n^5 + 154n^4 + 560n^3 + 784n^2 + 5040n - 14400)$$

Formula $QF_n^{(8)}$

$$(8) \quad QF^{(8)}(f) := \sum_{j=0}^{6} A_j^{(8)}(n) f\left(\frac{j}{n}\right) + A_7^{(8)}(n) \sum_{k\geq 7} f\left(\frac{k}{n}\right)$$

with

$$d_8 := (n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n^2-1)$$

$$A_0^{(8)}(n) = \frac{1}{5040d_8}(n+7)(1457n^6 + 32167n^5 + 327184n^4 + 2078378n^3 + 7281444n^2 + 1846800n - 47174400)$$

$$A_1^{(8)}(n) = \frac{1}{1680d_8}(2544n^7 + 64049n^6 + 616782n^5 + 3011179n^4 + 11335974n^3 + 47053872n^2 + 71578080n - 228614400)$$

$$A_2^{(8)}(n) = \frac{1}{1680d_8}(463n^7 + 16872n^6 + 182543n^5 + 164676n^4 - 5261646n^3 + 6425892n^2 + 132133680n - 228614400)$$

$$A_3^{(8)}(n) = \frac{1}{5040d_8}(9811n^7 + 270600n^6 + 3159697n^5 + 18213864n^4 + 37009372n^3 + 1337856n^2 + 340986240n - 685843200)$$

$$A_4^{(8)}(n) = \frac{1}{840d_8}(58n^7 + 1229n^6 + 38165n^5 + 706225n^4 + 3546057n^3 + 2596266n^2 + 59943240n - 114307200)$$

$$A_5^{(8)}(n) = \frac{1}{560d_8}(857n^7 + 21573n^6 + 206085n^5 + 1016235n^4 + 2742018n^3 + 1164672n^2 + 39402720n - 76204800)$$

$$A_6^{(8)}(n) = \frac{1}{8d_8}(7n^7 + 202n^6 + 2282n^5 + 12964n^4 + 38423n^3 + 18298n^2 + 564312n - 1088640)$$

$$A_7^{(8)}(n) = \frac{1}{d_8}(n^7 + 27n^6 + 294n^5 + 1638n^4 + 4809n^3 + 2268n^2 + 70524n - 136080)$$

Formula $QF_n^{(9)}$

$$(9)\quad QF_n^{(9)}(f) := \sum_{j=0}^{7} A_j^{(9)}(n) f\left(\frac{j}{n}\right) + A_8^{(8)}(n) \sum_{k\geq 8} f\left(\frac{k}{n}\right)$$

with

$$d_9 := (n+8)(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n^2-1)$$

$$A_0^{(9)} = \frac{1}{2160 d_9}(n+8)(565n^7 + 14806n^6 + 167381n^5 + 1284962n^4$$
$$+8599554n^3 + 38137212n^2 + 41759280n - 185068800)$$

$$A_1^{(9)} = \frac{1}{15120 d_9}(25504n^8 + 875217n^7 + 11918830n^6 + 78135435n^5$$
$$+240294166n^4 + 568660848n^3 + 3981566880n^2 + 10464249600n - 21337344000)$$

$$A_2^{(9)} = -\frac{1}{15120 d_9}(1793n^8 - 10080n^7 - 1701175n^6 - 22466220n^5 - 22624378n^4$$
$$+846138300n^3 + 596404080n^2 - 16741468800n + 21337344000)$$

$$A_3^{(9)} = \frac{1}{15120 d_9}(37193n^8 + 1236480n^7 + 17777795n^6 + 150150000n^5$$
$$+747265652n^4 + 1282669920n^3 - 1337160960n^2 + 14483750400n - 21337344000)$$

$$A_4^{(9)} = -\frac{1}{7560 d_9}(3998n^8 + 137235n^7 + 2138675n^6 + 14402535n^5$$
$$-7014553n^4 - 260126370n^3 + 290912040n^2 - 7713316800n + 10668672000)$$

$$A_5^{(9)} = \frac{1}{15120 d_9}(32915n^8 + 1126503n^7 + 15235367n^6 + 99829065n^5$$
$$+355563110n^4 + 727575072n^3 - 942911712n^2 + 15089276160n - 21337344000)$$

$$A_6^{(9)} = \frac{1}{840 d_9}(379n^8 + 16506n^7 + 305970n^6 + 2836260n^5$$
$$+14122731n^4 + 38210634n^3 - 46211320n^2 + 843259200n - 1185408000)$$

$$A_7^{(9)} = \frac{1}{9 d_9}(10n^8+335n^7+4744n^6+36470n^5+162106n^4+412160n^3-507384n^2+9025920n-12700800)$$

$$A_8^{(9)} = \frac{1}{d_9}(n^8 + 35n^7 + 510n^6 + 3990n^5 + 17913n^4 + 45780n^3 - 56260n^2 + 1002960n - 1411200)$$

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